JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **141**, 349–372 (1989)

# Contiguous Relations, Basic Hypergeometric Functions, and Orthogonal Polynomials, I

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*Submitted by George Gasper*

Received September 28, 1987

The second-order $q$-difference equation of a ${}_2\varphi_1$ and the contiguous relations of ${}_2\varphi_1$'s are used to derive continued fraction expansions for quotients of ${}_2\varphi_1$'s. We identify the cases when these continued fractions are positive definite $J$-fractions, i.e., the cases when the denominators of the approximants of the $J$-fractions are orthogonal polynomials. The measures with respect to which the denominator polynomials are orthogonal are found and the orthogonal polynomials are studied in some detail. 

## 1. INTRODUCTION

Gauss realized that one of the contiguous relations for the hypergeometric function can be used to derive a continued fraction expansion for a quotient of two hypergeometric functions. A reference to Gauss's work may be found in Perron [23], Wall [29], or Jones and Thron [20]. Later Heine used the same idea to find a continued fraction representation for a quotient of two basic hypergeometric functions (or Heine functions). E. Frank [14, 15, 16] continued Heine's investigations and obtained continued fraction expansions for other quotients of Heine series.

Orthogonal polynomials are denominators of approximants (or

* Research partially supported by a grant from the National Science Foundation.



0022-247X/89 $3.00

convergents) of positive definite $J$-fractions. Indeed when the positivity condition

$$A_n A_{n-1} C_n > 0, \qquad n = 1, 2, ..., \tag{1.1}$$

holds, the denominators of the approximants of the $J$-fraction

$$F(z) = \frac{A_0}{A_0 z + B_0 -} \frac{C_1}{A_1 z + B_1 -} \cdots \tag{1.2}$$

are orthogonal with respect to a positive measure $d\mu$. If, in addition, the boundedness conditions

$$|B_n/A_n| \leqslant M \qquad \text{and} \qquad C_n/(A_n A_{n-1}) \leqslant M, \qquad n = 1, 2, ... \tag{1.3}$$

also hold then

$$F(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t}, \qquad z \notin \operatorname{supp}\{d\mu\}. \tag{1.4}$$

This fact is called Markov's theorem (Shohat and Tamarkin [26], Szegö [28]). Once we find the Stieltjes transform $F(z)$ of $d\mu$, as in (1.4), we can compute $d\mu$ from the Perron–Stieltjes inversion formula

$$F(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t}$$

if and only if

$$\mu(t) - \mu(s) = \lim_{\varepsilon \to 0^+} \int_s^t \frac{F(u - i\varepsilon) - F(u + i\varepsilon)}{2\pi i}\, du. \tag{1.5}$$

In (1.5) it is assumed that $\mu(x)$ is normalized by $\mu(-\infty) = 0$, $2\mu(x) = [\mu(x+0) + \mu(x-0)]$.

Recently, several interesting measures were evaluated by determining the asymptotic behavior of the numerators and denominators of approximants of the corresponding $J$-fractions and then applying (1.5); see [7, 10–12, 18, 32].

This work started as an attempt to use the ideas of Gauss and Heine to replace the use of asymptotic methods in evaluating $J$-fractions. When we succeeded in our goal we decided to study systematically the contiguous relations and $q$-difference equations satisfied by basic hypergeometric functions. This paper deals only with ${}_2\varphi_1$ functions. We hope to treat the ${}_3\varphi_2$ and ${}_4\varphi_3$ functions in future papers. Using similar methods Ismail and Rahman, in an unfinished manuscript, found the measure with respect to which the associated Askey–Wilson polynomials are orthogonal. The latter polynomials provide a two-parameter generalization of the Wigner 6-$j$ symbols. Some authers refer to the Askey–Wilson polynomials as $q$-Wilson or ${}_4\varphi_3$ polynomials. The interested reader may consult [4, 8, 9 and 31] about the Askey–Wilson polynomials and connections with the 6-$j$ symbols, classical orthogonal polynomials, and Racah coefficients.

This paper is arranged as follows. In Section 2 we fix notation and state the results used in the subsequent sections. In Section 3 the second-order $q$-difference equation for a ${}_2\varphi_1$ is iterated, à la Gauss and Heine, to find a fairly general continued fraction. Special cases of that continued fraction include the $J$-fractions associated with the Al-Salam–Carlitz polynomials [1, 13, 18], the Al-Salam–Chihara polynomials [2, 7], and the associated $q$-ultraspherical polynomials [11], to name a few. It also includes the $J$-fraction of the associated $q$-Pollaczek polynomials, which is a new set of orthogonal polynomials. In Section 4 contiguous relations for ${}_2\varphi_1$'s are used to evaluate the $J$-fractions of two new sets of orthogonal polynomials. One of them is what we call the associated big $q$-Laguerre polynomials. They are a one-parameter generalization of a special case of the big $q$-Jacobi polynomials of Andrews and Askey [4]. In Section 5 we use the inversion formula (1.5) to find the orthogonality measures of the associated $q$-Pollaczek polynomials and identify the spectrum of the associated big $q$-Laguerre polynomials. We also give a new derivation of the orthogonality measure of the big $q$-Laguerre polynomials. In addition we find the absolutely continuous component of the measure of orthogonality of the new polynomials discovered in Section 3. In the next section, Section 6, we find a generating function of the associated big $q$-Laguerre polynomials of earlier sections. This generating function may be used to find explicit representations of the polynomials they generate. Our derivation of the generating function of the associated big $q$-Laguerre polynomials uses recursion relations and provides a new proof of a special case of a generating function of Ismail and Wilson [19]. It turned out that the associated big $q$-Laguerre polynomials are birth and death process polynomials. We discovered another family of associated big $q$-Laguerre polynomials arising from a slightly different birth and death process. In Section 7 we establish a generating function for the associated $q$-Pollaczek polynomials and use it to derive an asymptotic formula for these polynomials. We also prove the positivity of the linearization coefficients in products of associated $q$-Pollaczek polynomials in certain cases. The Pollaczek polynomials are treated in [24, 28].

A list of ${}_2\varphi_1$ contiguous relations is compiled in an appendix. Most of these relationships can be found in Heine's classical work [17] but do not seem to be accessible in the more recent literature. We made extensive use of these relations in this work. Wilson [30, 31] compiled a list of contiguous relations satisfied by the generalized hypergeometric functions ${}_3\varphi_2$ and ${}_4\varphi_3$.

## 2. Preliminaries

Given a number $q$, the $q$-shifted factorial may be defined by

$$\begin{aligned} (\sigma)_0 &:= 1, \qquad (\sigma)_n := \prod_{k=1}^{n} (1-\sigma q^{k-1}), \\ n &= 1, 2, ..., \qquad (\sigma)_{-n} := 1/(\sigma q^{-n})_n, \; n=0, 1, .... \end{aligned} \tag{2.1}$$

If $|q|<1$ we define

$$(\sigma)_\infty = \prod_{k=1}^{\infty} (1-\sigma q^{k-1}). \tag{2.2}$$

In other words $(\sigma)_n = (\sigma)_\infty/(\sigma q^n)_\infty$ for $n=0, \pm 1, \pm 2, ....$ A basic hypergeometric function ${}_{r+1}\varphi_r$ is defined by

$$\begin{aligned} &{}_{r+1}\varphi_r(a_1, ..., a_{r+1}; b_1, ..., b_r; q, z) \\ &\quad = \sum_{n=0}^{\infty} z^n \prod_{k=1}^{r+1} \left[\frac{(a_k)_n}{(b_k)_n}\right], \qquad b_{r+1} := q. \end{aligned} \tag{2.3}$$

The function

$$f(z) := {}_2\varphi_1(a, b; c; q, z) \tag{2.4}$$

satisfies the $q$-difference equation

$$(abz - c/q)\, f(q^2 z) - [(a+b)\, z - 1 - c/q]\, f(qz) = (1-z)\, f(z), \tag{2.5}$$

which will be used to evaluate a continued fraction in Section 3. The Heine transformation

$${}_2\varphi_1(a, b; c; q, z) = \frac{(az)_\infty (b)_\infty}{(z)_\infty (c)_\infty}\, {}_2\varphi_1(c/b, z; az; q, b) \tag{2.6}$$

and its iterate

$${}_2\varphi_1(a, b; c; q, z) = \frac{(bz)_\infty (c/b)_\infty}{(z)_\infty (c)_\infty}\, {}_2\varphi_1(abz/c, b; bz; q, c/b) \tag{2.7}$$

provide analytic continuation formulas. We shall use the $q$-binomial theorem [27, p. 246]

$$\frac{(cx)_\infty}{(x)_\infty} = \sum_{n=0}^{\infty} \frac{(c)_n}{(q)_n} x^n \tag{2.8}$$

and the Sears transformation

$$\begin{aligned}
&\frac{(AZ/C)_\infty}{(q/C)_\infty} {}_2\varphi_1(A, B; C; q, Z) \\
&\quad - \frac{(ABZ/C)_\infty}{(q/A)_\infty} {}_2\varphi_1(C/A, Cq/ABZ; qC/AZ; q, qB/C) \\
&= -\frac{(B)_\infty (C/A)_\infty (AZ/q)_\infty (q^2/AZ)_\infty}{(C/q)_\infty (qB/C)_\infty (q/A)_\infty (qC/Az)_\infty} {}_2\varphi_1(qA/C, qB/C; q^2/C; q, Z).
\end{aligned} \tag{2.9}$$

We found Theorem 2.1 below to be useful in proving convergence of continued fractions to a ratio of functions. The following definitions are used in the theorem. Following Jones and Thron [20], a series $L = c_m z^m + c_{m+1} z^{m+1} + c_{m+2} z^{m+2} + \ldots$, $c_m \neq 0$, is called a *formal Laurent series* (fLs), provided the $c_k$ are complex numbers and $m$ is an integer. Define $\lambda$ as follows: $\lambda(L) = m$ for $L \neq 0$ and $\lambda(0) = \infty$. If $f(z)$ is a function meromorphic at the origin, we will denote its Laurent expansion by $L(f)$. A sequence $\{R_n(z)\}$ of functions meromorphic at the origin will be said *to correspond to a fLs L* (at $z = 0$) if $\lim_{n \to \infty} \lambda(L - L(R_n)) = \infty$.

THEOREM 2.1. *Let $\{a_n(z)\}$ and $\{b_n(z)\}$ be sequences of functions meromorphic at the origin with $a_n(z)$ not identically zero, $n = 0, 1, 2, \ldots$. Assume that $\{s_n\}$ is a sequence of nonzero fLs satisfying the recurrence relations*

$$s_n = L(b_n)\, s_{n+1} + L(a_{n+1})\, s_{n+2}, \qquad n = 0, 1, 2, \ldots. \tag{2.10}$$

*Then for each $m = 0, 1, 2, \ldots$, the continued fraction*

$$b_m(z) + \frac{a_{m+1}(z)}{b_{m+1}(z)+} \frac{a_{m+2}(z)}{b_{m+2}(z)+} \cdots$$

*corresponds to the fLs $s_m/s_{m+1}$ provided that the following conditions are satisfied*:

$$\lambda(L(b_n)) + \lambda(L(b_{n-1})) < \lambda(L(a_n)), \qquad n = 1, 2, 3, \ldots, \tag{2.11}$$

$$\lambda(s_n/s_{n+1}) + \lambda(L(b_{n-1})) < \lambda(L(a_n)), \qquad n = 1, 2, 3, \ldots. \tag{2.12}$$

The above theorem is proved in Jones and Thron [20, pp. 160–161]. We shall use only the case $m=0$ of Theorem 2.1.

The numerator polynomials $\{N_n(x)\}$ and denominator polynomials $\{D_n(x)\}$ of the approximants of a $J$-fraction (1.2) satisfy the three-term recurrence relation

$$\begin{gathered} w_{n+1}(x)=(A_n x+B_n)\, w_n(x)-C_n w_{n-1}(x), \\ w_n(x)=N_n(x), \qquad \text{or} \qquad w_n(x)=D_n(x), \end{gathered} \tag{2.13}$$

and the initial conditions

$$N_0(x)=0, \qquad N_1(x)=A_0, \qquad D_0(x)=1, \qquad D_1(x)=A_0 x+B_0. \tag{2.14}$$

When the positivity condition (1.1) is satisfied both families are orthogonal with respect to positive measures with infinite support. Let $\{D_n(x)\}$ be orthogonal with respect to $d\mu$, and normalize $d\mu$ by $\int_{-\infty}^{\infty} d\mu=1$. This is the normalization in (1.2) and (1.4). The orthogonality relation is

$$\int_{-\infty}^{\infty} D_m(x)\, D_n(x)\, d\mu(x)=\zeta_n \delta_{mn}, \qquad \zeta_0=1, \zeta_n=A_0 C_1 C_2 \cdots C_n/A_n,\ n>0 \tag{2.15}$$

With this notation Markov's theorem says that if (1.1) and (1.3) hold then

$$\lim_{n\to\infty} \frac{N_n(x)}{D_n(x)}=\int_{-\infty}^{\infty} \frac{d\mu(t)}{x-t}, \qquad x \notin \operatorname{supp}\{d\mu\}. \tag{2.16}$$

Given a sequence of polynomials $\{D_n(x)\}$ recursively generated by (2.13) and (2.14), the polynomials $\{D_n^{(\gamma)}(x)\}$ generated by $D_0^{(\gamma)}(x)=1$, $D_1^{(\gamma)}(x)=A_\gamma x+B_\gamma$, and

$$D_{n+1}^{(\gamma)}(x)=(A_{n+\gamma} x+B_{n+\gamma})\, D_n^{(\gamma)}(x)-C_{n+\gamma} D_{n-1}^{(\gamma)}(x), \qquad n>0,$$

are called the associated polynomials of $\{D_n(x)\}$ of order $\gamma$, provided that $A_{n+\gamma}$ and $B_{n+\gamma}$ are well-defined, as is the case when $A_n$ and $B_n$ are quotients of polynomials or exponential polynomials.

Sometimes we may need to rescale coefficients in three-term recurrence relations (2.13). Two continued fractions are equivalent if they have the same sequence of $n$th convergents $N_n(x)/D_n(x)$. It is easy to see that if $g_n \neq 0$, then the following continued fractions have the same sequence of $n$th convergents and are equal:

$$b_0+\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots=b_0+\frac{g_1 a_1}{g_1 b_1+}\,\frac{g_1 g_2 a_2}{g_2 b_2\ +}\,\frac{g_2 g_3 a_3}{g_3 b_3\ +}\cdots. \tag{2.17}$$

The following theorem of Askey [5] will be used to prove that the linearization coefficients in the product of associated $q$-Pollaczek polynomials are nonnegative.

THEOREM 2.2. *Let $\{r_n(x)\}$ be a sequence of monic orthogonal polynomials, that is, $r_n(x)-x^n$ is a polynomial of degree at most $n-1$, and assume that*

$$r_1(x)\,r_n(x)=r_{n+1}(x)+a_n r_n(x)+b_n r_{n-1}(x), \qquad n\geqslant 0,\, b_{-1}:=0. \tag{2.18}$$

*If $a_n\geqslant 0$, $b_n>0$ and $a_{n+1}\geqslant a_n$, $b_{n+1}\geqslant b_n$, for all $n$, then*

$$r_m(x)\,r_n(x)=\sum_{k=|m-n|}^{m+n}\alpha_k(m,n)\,r_k(x) \tag{2.19}$$

*with $\alpha_k(m,n)\geqslant 0$ for all nonnegative integers $k$, $m$, $n$.*

Our results are stated for $|q|<1$ but in most cases one can reduce the case $|q|>1$ to $|q|<1$ by applying the transformation

$${}_2\varphi_1(1/a,\,1/b;\,1/c;\,1/q,\,abz/c)={}_2\varphi_1(a,\,b;\,c;\,q,\,qz). \tag{2.20}$$

Finally, $\pi(x)$ shall denote a generic polynomial in $x$ of degree at most 1.

## 3. $J$-FRACTION RESULTING FROM (2.5)

We first rewrite (2.5) in the form

$$\begin{aligned}(1-z)^{-1}&f(qz)/f(z)\\ &=[1-(a+b)\,z+c/q+(abz-c/q)f(q^2z)/f(qz)]^{-1},\end{aligned} \tag{3.1}$$

where $f$ is as in (2.4). We then iterate the functional relationship (3.1) and find

$$\frac{f(qz)}{(1-z)\,f(z)}=\frac{1}{b_0-}\,\frac{a_1}{b_1-}\cdots\frac{a_n}{b_n-}\,\frac{(c/q-abzq^n)\,f(zq^{n+2})}{f(zq^{n+1})}, \tag{3.2}$$

hence if these iterations converge we will obtain the continued fraction

$$\frac{f(qz)}{(1-z)\,f(z)}=\frac{1}{b_0-}\,\frac{a_1}{b_1-}\,\frac{a_2}{b_2-}\cdots, \qquad f(z)={}_2\varphi_1(a,\,b;\,c;\,q,\,z), \tag{3.3}$$

with

$$a_{n+1}=(cq^{-1}-abzq^n)(1-q^{n+1}z),$$
$$b_n=1+cq^{-1}-(a+b)\,zq^n, \qquad n\geqslant 0. \tag{3.4}$$

We now prove (3.3).

THEOREM 3.1. *Let $a_n$ and $b_n$ be as in* (3.4) *and assume that* $-1<q<1$. *The continued fraction on the right-hand side of* (3.3) *converges to the left-hand side provided that* $(a,b,c)$ *belongs to a neighborhood of* $(0,0,0)$, $|z|<1$, *and $z$ is not a pole of the right-hand side.*

*Proof.* Set $s_n=f(zq^n)$. We need to identify a variable $t$ that plays the role of the variable $z$ in Theorem 2.1. If $c\neq 0$ we replace $a, b, c$ by $at, bt, ct$, respectively. When $tc\neq q^{-n}$ for any nonnegative integer $n$ then $\lambda(a_n)=1$ but $\lambda(b_n)=0$. Furthermore $\lambda(s_n/s_{n+1})=0$ since ${}_2\varphi_1(0,0;0;q,z)=1/(z)_\infty$ and (2.11) and (2.12) hold. On the other hand, if $c=0$ we choose $z$ as our variable and we also find $\lambda(a_n)=1$, $\lambda(b_n)=0$, and $\lambda(s_n/s_{n+1})=0$ for $n\geqslant 0$. We then apply Theorem 2.1 and establish the required result, since fLs expansions of analytic function $(m=0)$ converge to their respective analytic functions in their domain of analyticity.

COROLLARY 3.2. *When $z=1$ in* (3.4) *the continued fraction in* (3.3) *converges to*

$$\frac{(q)_\infty (c)_\infty}{(a)_\infty (b)_\infty}\,{}_2\varphi_1(a,b;c;q,q). \tag{3.5}$$

*Proof.* As $z\to 1^-$ in (3.3), the right-hand side clearly converges. Abel's summation theorem shows that the left-hand side of (3.3) converges to the function in (3.5).

*Remark* 3.3. The left-hand side of (3.3) for general $a, b, c; c\neq q^{-n}$, $n=0,1,...;$ is an analytic continuation of the case $(a,b,c)$ in a neighborhood of $(0,0,0)$, as in Theorem 3.2. In the cases we are interested in we can analytically continue the continued fraction on the right-hand side of (3.3). The analytic continuations of both sides must agree on their common domain of analyticity.

Let $\{D_n\}$ be the denominators associated with (3.3) and (3.4), after using the rescaling formula (2.17) with $g_n=1/(1-zq^{n+1})$. The $D_n$'s satisfy

$$\begin{aligned}&(1-zq^{n+1})\,D_{n+1}\\ &\quad=(1+cq^{-1}-(a+b)\,zq^n)\,D_n-(cq^{-1}-abzq^{n-1})\,D_{n-1}.\end{aligned} \tag{3.6}$$

We now make choices for the parameters $a$, $b$, $c$, $z$ to reduce (3.6) to the form (2.13) and make the $D_n$'s multiples of orthogonal polynomials $p_n(x)$. Recall that $\pi(x)$ denotes a polynomial in $x$ of degree at most 1.

*Case* I. $c=0$. We set

$$a+b=\pi_1(x)/\pi_2(x) \qquad \text{and} \qquad ab=K/[\pi_2(x)]^2, \tag{3.7}$$

$K$ is a constant. The polynomials $p_n(x):=[\pi_2(x)]^n D_n$ turn out to be orthogonal polynomials of the variable $x$. They are generated by

$$p_0(x)=1, \qquad p_1(x)=[\pi_2(x)-\pi_1(x)\,z]/(1-zq), \tag{3.8}$$

and

$$\begin{aligned}(1-zq^{n+1})\,&p_{n+1}(x)\\ &=[\pi_2(x)-\pi_1(x)\,zq^n]\,p_n(x)+Kzq^{n-1}p_{n-1}(x), \qquad n>0.\end{aligned} \tag{3.9}$$

This corresponds to another rescaling with $g_n=\pi_2(x)$; see (2.17). In order for the positivity condition (1.1) to be satisfied it is necessary that $q>0$ and $Kz<0$ in (3.9). When $\pi_1(x)$ (or $\pi_2(x)$) is not a constant, there is no loss of generality in letting $\pi_1(x)$ (or $\pi_2(x)$) be $x$. The $q$-polynomials of Al-Salam and Carlitz [1, 13, 18] correspond to the case $\pi_2(x)=x$, $z=1$, $\pi_1(x)=1+K$, $K<0$. In general, when $z\neq 1$, $\pi_2(x)=x$, $\pi_1(x)=1+K$ we get the associated Al-Salam–Carlitz polynomials, which do not seem to have been studied before. For orthogonality $K$ must be positive if $z$ is negative. On the other hand, if $z>0$ then $K<0$ and $0<z<q^{-1}$. When $\pi_2(x)=x$, $\pi_1(x)=(1+K)x$ the $p_n(x)$'s generalize the $q$-analogue of the Carlitz–Karlin–McGregor polynomials; see [7, Chap. 7; 18]. The only remaining possibility is the case when $\pi_1(x)=x$ and $\pi_2(x)=Ax+B$. These polynomials generalize certain cases of the Al-Salam–Chihara polynomials [2, 7, 9], and the case $z=1$ was discovered in [18]. The spectrum of the polynomials under consideration is bounded and purely discrete; see Krein's theorem in [13]. Thus the continued fraction in (3.3) converges uniformly on compact subsets of the complex plane which do not contain the countably many spectral points. For completeness and future reference we record the continued fraction in this case. We have

$$\begin{aligned}&\frac{(\pi_2'(x)-z\pi_1'(x))\,{}_2\varphi_1(a,b;0;q,qz)}{(1-z)\,\pi_2(x)\,{}_2\varphi_1(a,b;0;q,z)}\\ &\qquad=\frac{A_0}{A_0x+B_0-}\,\frac{C_1}{A_1x+B_1-}\cdots\frac{C_n}{A_nx+B_n-}\cdots,\end{aligned} \tag{3.10}$$

with

$$A_nx+B_n=[\pi_2(x)-\pi_1(x)\,zq^n]/(1-zq^{n+1})$$

and

$$C_n = Kzq^{n-1}/(1-zq^{n+1}), \tag{3.11}$$

and $a$ and $b$ as in (3.7). The methods in [18] can be used to derive (3.10).

*Case* II. $c \neq 0$. We choose

$$c = qe^{-2i\theta}, \qquad ab = Ke^{-2i\theta}, \qquad x = \cos\theta, \qquad a+b = \pi_1(x)\, e^{-i\theta}. \tag{3.12}$$

It is assumed that $|e^{-i\theta}| < 1$, i.e., $x$ lies in the complex plane cut along $[-1, +1]$. In (3.6) we let $\rho_n(x) = e^{in\theta} D_n$, i.e., we rescale using $g_n = e^{i\theta}$. The $\rho_n(x)$'s satisfy

$$(1-zq^{n+1})\, \rho_{n+1}(x) = [2x - \pi_1(x)\, zq^n]\, \rho_n(x) - (1-zKq^{n-1})\, \rho_{n-1}(x). \tag{3.13}$$

These are the associated $q$-Pollaczek polynomials if $z \neq 1$. It is more convenient to use the parameters $z$, $U$, $V$, and $\Delta$, where

$$\pi_1(x) = 2(xU\Delta - V) \qquad \text{and} \qquad K = \Delta^2, \tag{3.14}$$

as in [12]. It is worth recording the continued fraction in this case.

THEOREM 3.4. *Let* $|q| \neq 1$, $|z| < 1$, $z\Delta^2 < 1$, $U\Delta z < 1$ *and*

$$A_n x + B_n = 2[x(1-U\Delta zq^n) + Vzq^n]/(1-zq^{n+1})$$

*and*

$$C_n = (1-z\Delta^2 q^{n-1})/(1-zq^{n+1}) \tag{3.15}$$

*hold. Then the continued fraction on the right-hand side of* (1.2) *converges to*

$$\begin{aligned} F(x) = {} & 2e^{-i\theta}(1-U\Delta z)(1-z)^{-1} \\ & \times {}_2\varphi_1(a, b; qe^{-2i\theta}; q, zq)/{}_2\varphi_1(a, b; qe^{-2i\theta}; q, z), \end{aligned} \tag{3.16}$$

*in the complex x-plane cut along* $[-1, 1]$ *except at the zeros of* ${}_2\varphi_1$ *in the denominator of* (3.16), *when a and b are as in* (3.12) *and* (3.14).

*Proof.* We need only to discuss the analytic continuation from $(a, b, c)$ in a neighborhood of the origin $(0, 0, 0)$ to the case stated in the theorem. We made the choices $c = qe^{-2i\theta}$. Since $e^{-2i\theta} \to 0$ as $x \to \infty$ in the complex $x$-plane, $x = \cos\theta$, the continued fraction in (3.3) must converge if $x$ is not real. This shows that the spectrum is bounded. The continuous spectrum is $[-1, 1]$ since the left-hand side of (3.3) is single-valued across the set $(-\infty, 1] \cup [1, \infty)$. This completes the proof.

Note that $K$, $U$, $V$ and $\Delta$ do not depend on $x$ but $a$ and $b$ depend on $x$ through (3.12).

In the case $z=1$, which gives rise to the $q$-Pollaczek polynomials, the measure was found in [12]. See [12] for additional properties of the Pollaczek polynomials and their $q$-analogues. The $q$-Pollaczek polynomials arose later in the solution, by Al-Salam and Chihara [3], of a conjecture of Andrews and Askey. The $q$-ultraspherical polynomials of L. J. Rogers [6, 9] correspond to $z=1$, $\Delta=\beta$, $U=1$, and $V=0$.

## 4. Contiguous Relations

In [15] E. Frank proved the following theorem.

THEOREM 4.1. *If* $|q|\neq 1$, *the continued fraction in* (4.1),

$$\frac{{}_2\varphi_1(a,b;c;q,z)}{{}_2\varphi_1(aq,bq;cq;q,z)}=b_0+\frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots, \tag{4.1}$$

*with*

$$\begin{aligned} a_{n+1}&=\frac{(1-aq^{n+1})(1-bq^{n+1})(c-abzq^{n+1})\,zq^n}{(1-cq^n)(1-cq^{n+1})},\\ b_n&=1-\frac{[a+b-abq^n(1+q)]\,zq^n}{1-cq^n}, \qquad n=0,1,..., \end{aligned} \tag{4.2}$$

*represents a meromorphic function of z which is equal to the function*

$${}_2\varphi_1(a,b;c;q,z)/{}_2\varphi_1(aq,bq;cq;q,z)$$

*throughout the finite z-plane except at zeros of* ${}_2\varphi_1(aq,bq;cq;q,z)$. *It is equal to this ratio in a neighborhood of* $z=0$ *and furnishes the analytic continuation of it throughout the finite z-plane.*

We now rescale the $a_n$'s and the $b_n$'s using (2.17) with

$$g_n=z^{-1}(1-cq^n)/[(1-aq^{n+1})(1-bq^{n+1})].$$

This establishes (4.1) with $a_n$'s and $b_n$'s given by

$$\begin{aligned} &(1-aq^{n+2})(1-bq^{n+2})\,a_{n+1}=-(s+abq^{n+1})\,q^n, \qquad n\geqslant 0,\\ &(1-aq^{n+1})(1-bq^{n+1})\,b_n\\ &\qquad=z^{-1}+q^n\{s-a-b+ab(1+q)\,q^n\}, \qquad n\geqslant 0,\ s:=-c/z. \end{aligned}$$

We now choose $z = -1/x$. This leads to a new set of orthogonal polynomials which we shall denote by $\{P_n(x; a, b, s)\}$. They are generated by

$$P_0(x; a, b, s) = 1,$$

$$P_1(x; a, b, s) = [-x + (s - a - b) + ab(1+q)]/[(1-aq)(1-bq)] \tag{4.3a}$$

$$\begin{aligned}(1-aq^{n+1})(1-bq^{n+1})\, P_{n+1}(x; a, b, s) \\ = [-x + q^n(s-a-b) + ab(1+q)\, q^{2n}]\, P_n(x; a, b, s) \\ - (s + abq^n)\, q^{n-1} P_{n-1}(x; a, b, s), \qquad n > 0,\end{aligned} \tag{4.3b}$$

and their continued fraction is a multiple of the reciprocal of the continued fraction in (4.1). We shall refer to these polynomials as associated big $q$-Laguerre polynomials. The big $q$-Laguerre polynomials are $\{P_n(x; a, 1, s)\}$. Observe that $\{P_n(x; a, b, s)\}$ are birth and death process polynomials [7, 18]. Before we study these polynomials we apply the Heine transformation (2.6) to analytically continue the ${}_2\varphi_1$'s in (4.1). The result is the following theorem.

THEOREM 4.2. *If* $0 < q < 1$, *and* $\{Q_n(x; a, b, s)\}$ *satisfy* (4.3b) *and the initial conditions*

$$Q_0(x; a, b, s) = 0, \qquad Q_1(x; a, b, s) = -[(1-aq)(1-bq)]^{-1},$$

*and if* $-1 < a, b < 1$, *and* $s > 0$, *then the limit*

$$\lim_{n\to\infty} \frac{Q_n(x; a, b, s)}{P_n(x; a, b, s)} = \frac{{}_2\varphi_1(s/bx, -1/x; -aq/x, q, bq)}{(x+a)(1-b)\, {}_2\varphi_1(s/bx, -1/x; -a/x; q, b)} \tag{4.4}$$

*holds uniformly in x on compact subsets of the complex plane not containing the origin nor any pole of the right-hand side in* (4.4).

The ${}_2\varphi_1$ in the denominator of (4.4) can be summed when $b = 1$ by the Abel method of summation. By symmetry it can also be summed when $a = 1$. Thus Corollary 4.3 holds.

COROLLARY 4.3. *Let* $0 < q < 1, -1 < a < 1$, *and* $s > 0$, *and* $Q_n$ *be as in Theorem* 4.2. *Then*

$$\begin{aligned}&\lim_{n\to\infty} \frac{Q_n(x; a, 1, s)}{P_n(x; a, 1, s)} \\ &\qquad = \frac{(q)_\infty(-aq/x)_\infty}{x(s/x)_\infty(-1/x)_\infty}\, {}_2\varphi_1(s/x, -1/x; -aq/x; q, q)\end{aligned} \tag{4.5}$$

*holds if* $x \neq -q^n, sq^n, n=0, 1, \ldots$. *The limit is uniform on compact subsets of the complex x-plane containing no member of the set* $\{x \mid x=-q^n$ *or* $x=sq^n, n=0, 1, \ldots\}$.

Theorem 4.1 can be proved by iterating the contiguous relation (A.18) and applying Theorem 2.1. It is natural to wonder whether any of the remaining contiguous relations lead to orthogonal polynomials. It is obvious that (A.2) and (A.10) can be iterated and would lead to continued fraction expansions for ${}_2\varphi_1(a, b; c, q, z)/{}_2\varphi_1(aq, b; c, q, z)$ and ${}_2\varphi_1(a, b; c, q, z)/{}_2\varphi_1(a, b; cq, q, z)$. The equivalence of these two continued fractions can be proved using the Heine transformation (2.6). One can use (A.14) to prove the equivalence of the continued fractions (4.1) and the one representing the quotient ${}_2\varphi_1(a, b; c, q, z)/{}_2\varphi_1(aq, b; c, q, z)$. We then conclude that this procedure does not seem to lead to any other orthogonal polynomials.

## 5. Measures

We first find the measure $d\mu$ with respect to which the associated $q$-Pollaczek polynomials are orthogonal. Let $F(x)$ be as in (3.16). The support of the discrete part of the measure coincides with the closure of the set of the zeros of ${}_2\varphi_1(a, b; qe^{-2i\theta}; q, z)$, with $a$ and $b$ as in (3.12). It is clear from the inversion formula (1.5) that $\pi\mu'$ is the imaginary part of $F(x-i0^+)$ and that $\mu'$ will be supported on the closure of the set $S$ of real points $x$ where $F(x)$ is not single valued across $S$. Observe that the products $(a)_n(b)_n$ in the ${}_2\varphi_1$'s in (3.16) are polynomials in $x$, hence are single-valued functions of $x$. Thus in order to find $\mu'$ explicitly we need to simplify the expression

$$\begin{aligned} G(z) := e^{-i\theta}\, {}_2\varphi_1(a, b; qe^{-2i\theta}; q, qz)\, {}_2\varphi_1(a, b; qe^{2i\theta}; q, z) \\ - \text{its complex conjugate.} \end{aligned}$$

We use (2.5), (3.12), and (3.14) to discover the functional equation

$$G(qz) = (1-z)(1-\Delta^2 z)^{-1}\, G(z).$$

Iterating the above functional equation and taking into account that $G(z) \to G(0) = -2i \sin\theta$, as $z \to 0$ we obtain

$$G(z) = -2i \sin\theta (\Delta^2 z)_\infty/(z)_\infty.$$

The above analysis yields

$$\mu'(x) = \frac{2(1 - Uz\Delta)(z\Delta^2)_\infty \sin\theta}{\pi(1-z)(z)_\infty} |{}_2\varphi_1(a, b; qe^{-2i\theta}; q, z)|^{-2}, \tag{5.1}$$

where $x = \cos\theta$ and $a$ as well as $b$ depends on $x$ as in (3.12) and (3.14).

To illustrate the power of this method we show how to compute the measure of the big $q$-Laguerre polynomials. Andrews and Askey used ingenious methods to guess what the measure should be, then they used their deep knowledge and understanding of the theory of basic hypergeometric functions to prove their guess.

The poles of the right-hand side of (4.5) are at $x = x_{n,1} := -q^n$ or $x = x_{n,2} := sq^n$. Let $M_{n,j}$ be the mass located at $x_{n,j}$. Clearly $M_{n,j}$ is the residue of the function on the right-hand side of (4.5). The $q$-analogue of the Vandermonde sum is [27, p. 247]

$${}_2\varphi_1(q^{-n}, A; B; q, q) = (B/A)_n A^n/(B)_n. \tag{5.2}$$

A calculation using (5.2) gives

$$M_{n,1} = \frac{(aq)_\infty(-aq/s)_n}{(-s)_\infty(q)_n(-q/s)_n} q^n, \qquad M_{n,2} = \frac{(aq)_n(-aq/s)_\infty}{(-1/s)_\infty(q)_n(-qs)_n} q^n. \tag{5.3}$$

Note that the polynomials $P_n$ and $Q_n$ in (4.5) follow the normalization of $D_n$ and $N_n$ in (2.13) and (2.14), hence the corresponding measure is a probability measure. It is clear that the point $x = 0$ is an essential singularity of both sides of (4.5). One can show that $x = 0$ in not a mass point in two different ways. The first is to prove directly that $\sum_{j=1}^2 \sum_{n=0}^\infty M_{n,j} = 1$. The second way is to appeal to Corollary 2.6 on pages 45–46 in [26], which says that in a determinate moment problem a point $x = \zeta$ supports a mass if and only if the sum of the squares of the orthonormal polynomials converges at $x = \zeta$. To apply the first approach note that

$$\sum_{j=1}^2 \sum_{n=0}^\infty M_{n,j} = \frac{(aq)_\infty}{(-s)_\infty} {}_2\varphi_1(-aq/s, 0; -q/s; q, q) + \frac{(-aq/s)_\infty}{(-1/s)_\infty} {}_2\varphi_1(aq, 0; -sq; q, q). \tag{5.4}$$

When we let $Z = q$ in (2.9) the second term on the left-hand side of (2.9) can be summed using the $q$-binomial theorem (2.8). We then let $B \to 0$ in the resulting identity and choose $A = -aq/s$, $C = -q/s$. A simple calculation shows that the right-hand side of (5.4) is 1, hence $x = 0$ does not support a point mass. In order to apply the second method we use (2.13), (2.15), (4.3a), and (4.3b) to identify the orthonormal polynomials as

$$\omega_n(x) := s^{-n/2} q^{-n(n-1)/4} \sqrt{\frac{(aq)_n (q)_n}{(-qa/s)_n}}\, P_n(x; a, 1, s).$$

If $\Sigma(\omega_n(0))^2$ converged then $P_n(0; a, 1, s) \to 0$ as $n \to \infty$ and by the regularity of Abel summability $(1-t)^{-1} \Sigma t^n P_n(0; a, 1, s)$ would converge to zero as $t \to 1^-$, in contradiction with the generating function (6.12).

## 6. Associated Big $q$-Laguerre Polynomials

To derive a generating function for the big $q$-Laguerre polynomials we let

$$L(x, t) = \sum_{n=0}^{\infty} P_n(x; a, b, s)\, t^n \tag{6.1}$$

be a generating function of the associated big $q$-Laguerre polynomials. Multiply (4.3b) by $t^n$ and add the resulting equalities to get

$$\begin{aligned}(1+xt)\, L(x, t) - (1-t)(a+b+st)\, L(x, qt)& \\ + ab(1-t)(1-qt)\, L(x, q^2 t) &= (1-a)(1-b).\end{aligned} \tag{6.2}$$

If $H(x, t)$ is a solution of the homogeneous $q$-difference equation associated with (6.2) then

$$\frac{(1-t)\, H(x, qt)}{(1+xt)\, H(x, t)} = \frac{1}{a+b+st - ab(1+qxt) \dfrac{(1-qt)\, H(x, q^2 t)}{(1+qxt)\, H(x, qt)}}. \tag{6.3}$$

We can then iterate (6.3) and if the iterations converge we will find

$$\begin{aligned}&\frac{(1-t)\, H(x, qt)}{(1+xt)\, H(x, t)} \\ &\quad = \frac{1}{a+b+st-}\, \frac{ab(1+qxt)}{a+b+sqt-} \cdots \frac{ab(1+xq^n t)}{a+b+sq^n t-} \cdots.\end{aligned} \tag{6.4}$$

The formal relationship (6.4) gives us a clue that $H(x, t)$ is of the form $f(qt)/f(t)$, where $f(t)$ is a power series in $t$. This suggests that we set

$$L(x, t) = (t)_\infty\, f(t)/(-xt)_\infty \tag{6.5}$$

and proceed to find $f(t)$ from (6.2). Thus we have

$$\begin{aligned} f(t)-(a+b+st)\,f(qt)+ab(1+xtq)\,f(qt^2) \\ =(1-a)(1-b)(-qxt)_\infty/(t)_\infty. \end{aligned} \tag{6.6}$$

The substitution $f(t)=\sum_{n=0}^{\infty} f_n t^n$ in (6.8) and equating coefficients of like powers of $t$ imply the inhomogeneous two-term recursion

$$\begin{aligned} &(1-aq^n)(1-bq^n)\,f_n \\ &\quad=q^{n-1}(s-abxq^n)\,f_{n-1}+(1-a)(1-b)(-qx)_n/(q)_n, \end{aligned} \tag{6.7}$$

where we used the $q$-binomial theorem (2.8)

$$\frac{(cx)_\infty}{(x)_\infty}=\sum_{n=0}^{\infty}\frac{(c)_n}{(q)_n}x^n. \tag{6.8}$$

In order to solve (6.7) we first write it as

$$\begin{aligned} &\frac{(aq)_n(bq)_n}{(qabx/s)_n\,s^n q^{n(n-1)/2}}f_n \\ &\quad=\frac{(aq)_{n-1}(bq)_{n-1}}{(qabx)_{n-1}\,s^{n-1}q^{(n-1)(n-2)/2}}f_{n-1}+\frac{(a)_n(b)_n(-qx)_n/(q)_n}{(qabx/s)_n\,s^n q^{n(n-1)/2}}. \end{aligned} \tag{6.9}$$

The general solution of the difference equation (6.9) is clearly

$$f_n=\frac{(qabx/s)_n\,s^n}{(aq)_n(bq)_n}q^{n(n-1)/2}\left[A+\sum_{j=0}^{n}\frac{(a)_j(b)_j(-qx)_j}{(qabx/s)_j(q)_j}s^{-j}q^{-j(j-1)/2}\right], \tag{6.10}$$

where $A$ is a constant. If $a=1$ or $b=1$ then $A=1$, otherwise $A=0$. Therefore we proved the generating relations

$$\begin{aligned} \sum_{n=0}^{\infty}P_n(x;a,b,s)t^n=\frac{(t)_\infty}{(-xt)_\infty}\sum_{n,j\geq 0}\frac{(qabx/s)_{n+j}(a)_j(b)_j(-qx)_j}{(aq)_{n+j}(bq)_{n+j}(qabx/s)_j(q)_j} \\ \times q^{nj+n(n-1)/2}t^{n+j}s^n, \qquad a\neq 1,\, b\neq 1, \end{aligned} \tag{6.11}$$

and

$$\sum_{n=0}^{\infty}P_n(x;a,1,s)t^n=\frac{(t)_\infty}{(-xt)_\infty}\sum_{n=0}^{\infty}\frac{(qax/s)_n}{(qa)_n(q)_n}q^{n(n-1)/2}(st)^n. \tag{6.12}$$

A generating function for the big $q$-Jacobi polynomials was derived in [19] using a transformation for a ${}_3\varphi_2$ and the explicit representation of a big $q$-Jacobi polynomial as a ${}_3\varphi_2$ function. Since $\{P_n(x;1,b,s)\}$ are special big $q$-Jacobi polynomials then (6.12) will follow from [19]. Even in the case $a=1$ or $b=1$ the above proof is more elementary than the proof in [19]. Furthermore we get the explicit representation of a big $q$-Laguerre polynomial as a consequence of (6.12). It is clear that by equating coefficients of similar powers of $t$ in (6.11) one can derive an explicit representation of $P_n(x; a, b, s)$ as a double sum.

Recall that a birth and death process with birth rates $\{\lambda_n\}$ and death rates $\{\mu_n\}$ generates a sequence of polynomials orthogonal with respect to the spectral measure of the process; see [7] for references and examples. It is assumed that $\lambda_n$ and $\mu_{n+1}$ are positive numbers for $n \geqslant 0$ but $\mu_0 \geqslant 0$. The orthogonal polynomials $\{p_n(x)\}$ are recursively generated by

$$\begin{aligned} &p_0(x)=1; \qquad p_1(x)=(-x+\lambda_0+\mu_0)/\mu_1; \\ &-xp_n(x)=\mu_{n+1}\,p_{n+1}(x)+\lambda_{n-1}\,p_{n-1}(x)-(\lambda_n+\mu_n)\,p_n(x). \end{aligned} \tag{6.13}$$

The Laguerre polynomials correspond to $\lambda_n = n+\alpha$ and $\mu_n = n$. The corresponding polynomials $\{P_n(x+1; a, b, s)\}$ correspond to $\mu_n = (1-aq^n)(1-bq^n)$ and $\lambda_n = q^n(s+abq^{n+1})$, $n \geqslant 0$. There is another family of associated big $q$-Laguerre polynomials which corresponds to the choices

$$\begin{aligned} &\mu_{n+1}=(1-aq^{n+1})(1-bq^{n+1}), \\ &\lambda_n = q^n(s+abq^{n+1}) \qquad \text{for} \quad n \geqslant 0 \text{ and } \mu_0 = 0. \end{aligned} \tag{6.14}$$

We shall denote the latter polynomials by $\{\mathscr{P}_n(x+1; a, b, s)\}$. They are defined via

$$\begin{aligned} &\mathscr{P}_0(x; a, b, s)=1, \\ &\mathscr{P}_1(x; a, b, s)=[-x+1+s+abq]/[(1-aq)(1-bq)] \end{aligned} \tag{6.15a}$$

$$\begin{aligned} &(1-aq^{n+1})(1-bq^{n+1})\,\mathscr{P}_{n+1}(x; a, b, s) \\ &\quad =[-x+q^n(s-a-b)+ab(1+q)\,q^{2n}]\,\mathscr{P}_n(x; a, b, s) \\ &\qquad -(s+abq^n)\,q^{n-1}\mathscr{P}_{n-1}(x; a, b, s), \qquad n>0. \end{aligned} \tag{6.15b}$$

As before we set

$$\mathscr{L}(x, t)=\sum_{n=0}^{\infty} \mathscr{P}_n(x; a, b, s)\,t^n, \tag{6.16}$$

which transforms the recurrence relation (6.15b) into the $q$-difference equation

$$\begin{aligned} &(1+xt)\,\mathscr{L}(x, t)-(1-t)(a+b+st)\,\mathscr{L}(x, qt) \\ &\quad +ab(1-t)(1-qt)\,\mathscr{L}(x, q^2t)=(1-a)(1-b)(1-t). \end{aligned} \tag{6.17}$$

Here again we define an auxiliary function $f(t)$ by $\mathscr{L}(x,t)=(t)_\infty f(t)/(-xt)_\infty$ with $f(t)=\sum_{n\geq 0} f_n t^n$. The $f_n$'s satisfy

$$\frac{(aq)_n(bq)_n}{(qabx/s)_n\, s^n q^{n(n-1)/2}} f_n$$
$$=\frac{(aq)_{n-1}(bq)_{n-1}}{(qabx/s)_{n-1}\, s^{n-1}q^{(n-1)(n-2)/2}} f_{n-1}+\frac{(a)_n(b)_n(-x)_n/(q)_n}{(qabx/s)_n\, s^n q^{n(n-3)/2}},$$

which leads to the generating function

$$\sum_{n=0}^{\infty} \mathscr{P}_n(x;a,b,s)t^n=\frac{(t)_\infty}{(-xt)_\infty}\sum_{n,j\geq 0}\frac{(qabx/s)_{n+j}(a)_j(b)_j(-x)_j}{(aq)_{n+j}(bq)_{n+j}(qabx/s)_j(q)_j}$$
$$\times q^{j+nj+n(n-1)/2}t^{n+j}s^n, \qquad a\neq 1, b\neq 1. \tag{6.18}$$

We conclude this section by computing the $J$-fraction associated with the polynomials $\{\mathscr{P}_n(x;a,b,s)\}$. Let $\{\mathscr{Q}_n(x;a,b,s)\}$ be their numerator polynomials $\{P_n(x;a,b,s)\}$ and $\{Q_n(x;a,b,s)\}$ are linear independent solutions of the second-order difference equation (6.15b) then both $\mathscr{P}_n(x;a,b,s)$ and $\mathscr{Q}_n(x;a,b,s)$ are linear combinations of $P_n(x;a,b,s)$ and $Q_n(x;a,b,s)$. The coefficients in these linear combinations can be found from the initial conditions (4.3a), (6.15a) and $\mathscr{Q}_0(x;a,b,s)=0$, $\mathscr{Q}_1(x;a,b,s)=-[(1-aq)(1-bq)]^{-1}$. Therefore

$$\mathscr{Q}_n(x;a,b,s)=Q_n(x;a,b,s),$$
$$\mathscr{P}_n(x;a,b,s)=P_n(x;a,b,s)+(1-a)(1-b)\,Q_n(x;a,b,s). \tag{6.19}$$

THEOREM 6.1. *We have*

$$\lim_{n\to\infty}\frac{\mathscr{Q}_n(x;a,b,s)}{\mathscr{P}_n(x;a,b,s)}=\frac{{}_2\varphi_1(-s/bx,-1/x;-aq/x;q,qb)}{(1-b)x\;{}_2\varphi_1(s/bx,-q/x;-aq/x;q,b)}. \tag{6.20}$$

*Furthermore the limit holds uniformly on compact subsets of the complex $x$-plane not containing the origin or any poles of the right-hand side of* (6.20).

*Proof.* Formula (6.19) and Theorem 4.2 show that $\mathscr{Q}_n(x;a,b,s)/\mathscr{P}_n(x;a,b,s)$ converges to $F(x)/[1+(1-a)(1-b)F(x)]$, where $F(x)$ is the right-hand side of (4.4). To simplify the quantity $1+(1-a)(1-b)F(x)$ we consult the list of contiguous relations to express

$$(1-b)(x+a)\;{}_2\varphi_1(s/(bx),-1/x;\;-a/x;q,b)$$
$$+(1-a)(1-b)\;{}_2\varphi_1(s/(bx),-1/x;-aq/x;q,qb) \tag{6.21}$$

as a ${}_2\varphi_1$ function. The appropriate contiguous relation is use is (A.27) and the expression (6.21) is $(1-b)x\, {}_2\varphi_1(s/(bx), -q/x; -aq/x; q, b)$ and the proof is complete.

## 7. Associated $q$-Pollaczek Polynomials

We first find a generating function for the associated $q$-Pollaczek polynomials. Let

$$R(x, t) = \sum_{n=0}^{\infty} \rho_n(x) t^n, \tag{7.1}$$

where the $\rho_n(x)$'s are associated $q$-Pollaczek polynomials of (3.13) and (3.14). We multiply (3.13) by $t^n$ and add for $n \geqslant 0$. This gives the $q$-difference equation

$$(1-2xt+t^2)\, R(x, t) = 1 - z + z\{1 - 2t(xU\Delta - V)t + \Delta^2 t^2\}\, R(x, qt),$$

which can be iterated to obtain

$$R(x, t) = \frac{(1-z)}{(1-te^{i\theta})(1-te^{-i\theta})}\, {}_3\varphi_2(ate^{i\theta}, bte^{i\theta}, q; qte^{i\theta}, qte^{-i\theta}; q, z), \tag{7.2}$$

from (3.12) and (3.14). One can use (7.2) to derive explicit representations of the polynomials. See Section 3 in [7] or [11] for examples of this calculation. The application of the asymptotic method of Darboux [22, 28] to (7.2) leads to the asymptotic formula

$$\rho_n(\cos\theta) \approx (1-z)(1-e^{-2i\theta})^{-1}\, e^{in\theta}\, {}_2\varphi_1(a, b; qe^{-2i\theta}; q, z)$$
$$+ \text{its complex conjugate}, \qquad 0 < \theta < \pi.$$

Letting

$$\phi := \arg({}_2\varphi_1(a, b; qe^{-2i\theta}; q, z)),$$

the above asymptotic formula takes the form

$$\rho_n(\cos\theta) \approx (1-z)\csc\theta\, |{}_2\varphi_1(a, b; qe^{-2i\theta}; q, z)|\, \cos[(n+1)\,\theta + \phi - \pi/2]. \tag{7.3}$$

One can use (7.3), Corollaries 36 (p. 141), and Theorem 40 (p. 143) in [21] to find the weight function $\mu'(x)$ of (5.1). For similar derivations we refer the interested reader to Chapters 3 and 7 in Askey and Ismail [7]. See also [11].

We now prove the following positivity result.

THEOREM 7.1. *The coefficients in the linearization of a product of two associated Pollaczek polynomials are nonnegative when the following assumptions are fulfilled*:

(i) $q, z, U, \Delta \in (0, 1)$, $V \geqslant 0$.

(ii) $U\Delta^2 - (1+q)\Delta + qU > 0$.

(iii) $z\Delta$ *does not exceed the positive zero of the polynomial* $h(x)$,

$$h(x) = qU[U\Delta^2 - (1+q)\Delta + qU]\, x^2 + 2q(1-U^2)\, x - q - \Delta^2 + \Delta U(1+q).$$

*Proof.* We shall show that the assumptions in Theorem 2.2 are satisfied. Recall that $\rho_1(x) = 2[x(1-U\Delta z) + Vz]/(1-qz)$. The corresponding monic polynomials $\{r_n(x)\}$ are defined by $r_n(x) = 2^{-n}(qz)_n\, \rho_n(x)/(zU\Delta)_n$. In this case the $a_n$'s and $b_n$'s in (2.18) are

$$a_n = Vz\left\{\frac{1}{1-zU\Delta} - \frac{q^n}{1-zU\Delta q^n}\right\}, \qquad b_n = \frac{(1-z\Delta^2 q^{n-1})(1-zq^n)}{4(1-zU\Delta q^{n-1})(1-zU\Delta q^n)}. \tag{7.4}$$

The nonnegativity of $a_n$ and $a_{n+1} - a_n$ is now obvious. The monotonicity of $b_n$ is slightly harder to prove. Let $y = zq^{n-1}$ in $b_n$. It is easy to see that

$$\frac{(1-y\Delta^2)(1-qy)}{(1-yU\Delta)(1-qyU\Delta)} = U^{-2} + \frac{A}{1-yU\Delta} + \frac{B}{1-yqU\Delta},$$

with

$$(1-q)\, U^2\Delta A = (q - U\Delta)(\Delta - U)$$

and

$$(1-q)\, U^2\Delta B = (1-U\Delta)(qU - \Delta).$$

Therefore, by (7.4), the $b_n$'s will form an increasing sequence if the derivative of the function

$$g(y) = A(1-yU\Delta)^{-1} + B(1-yqU\Delta)^{-1}$$

is not positive on $(0, z]$. A calculation shows that $g'(x)$ is a positive multiple of $h(yU)$ and $h$ is the polynomial in (iii). This completes the proof.

## APPENDIX: A LIST OF CONTIGUOUS RELATIONS FOR ${}_2\varphi_1$'S

To save space we use the following notations. The symbol $\Phi$ shall always mean ${}_2\varphi_1(a, b; c; q, z)$. In analogy with the notation used in the theory of hypergeometric functions (see, for example, Rainville [25]), we use $\Phi(a\pm)$, $\Phi(b\pm)$, or $\Phi(c\pm)$ to denote ${}_2\varphi_1(aq^{\pm 1}, b; c; q, z)$, ${}_2\varphi_1(a, bq^{\pm 1}; c; q, z)$, or ${}_2\varphi_1(a, b; cq^{\pm 1}; q, z)$. We also use $\Phi(qz)$ to mean ${}_2\varphi_1(a, b; c; q, qz)$.

The first equation is the $q$-difference equation:

$$(1-z)\,\Phi - [cq^{-1} + 1 - z(a+b)]\,\Phi(qz) + (cq^{-1} - abz)\,\Phi(q^2z) = 0. \tag{A.1}$$

We start by giving a complete list of the contiguous relations, excluding those obtained by permuting parameters.

$$[c(1+q-a) - aq + az(a-b)]\,\Phi$$
$$- (c-abz)(1-a)\,\Phi(a+) - q(c-a)\,\Phi(a-) = 0. \tag{A.2}$$

$$(a-b)\,\Phi + b(1-a)\,\Phi(a+) - a(1-b)\,\Phi(b+) = 0. \tag{A.3}$$

$$[ab(c+q) - c(aq+b)]\,\Phi$$
$$+ b(c-abz)(1-a)\,\Phi(a+) + aq(c-b)\,\Phi(b-) = 0. \tag{A.4}$$

$$(1-c)[c^2(1-a) + a^2z(c-b)]\,\Phi$$
$$- (c-abz)(1-a)(1-c)\,\Phi(a+) + az(c-a)(c-b)\,\Phi(c+) = 0. \tag{A.5}$$

$$(c-aq)\Phi - c(1-a)\,\Phi(a+) + a(q-c)\,\Phi(c-) = 0. \tag{A.6}$$

$$(cq-abz)(a-b)\Phi + bq(c-a)\,\Phi(a-) - aq(c-b)\,\Phi(b-) = 0. \tag{A.7}$$

$$(cq-abz)(1-c)\Phi + cq(1-c)\,\Phi(a-) + az(c-b)\,\Phi(c+) = 0. \tag{A.8}$$

$$[c^2(q-a) + a^2z(c-bq)]\Phi$$
$$- cq(c-a)\,\Phi(a-) - a(c-abz)(q-c)\,\Phi(c-) = 0. \tag{A.9}$$

$$(1-c)[c(q-c) + z\{c(a+b) - ab(1+q)\}]\Phi$$
$$+ z(c-a)(c-b)\,\Phi(c+) - (c-abz)(q-c)(1-c)\,\Phi(c-) = 0. \tag{A.10}$$

Next we give a list of relations where any parameter can be increased or decreased by one. The following notation will be used. Let $\Phi = \varphi$, $\varphi_1 = \Phi(a+)$; $\varphi_2 = {}_2\varphi_1(aq, bq; c, q, z)$; $\varphi_3 = {}_2\varphi_1(aq, bq; cq, q, z)$; $\varphi_4 = {}_2\varphi_1(aq^2, bq; cq, q, z)$; $\varphi_5 = {}_2\varphi_1(aq^2, bq^2; cq, q, z)$; $\varphi_6 = {}_2\varphi_1(aq^2, bq^2; cq^2, q, z)$.

$$bq(c-aq)\varphi - [cq(a+b) - abq(c+q)]\,\varphi_1 + aq(c-abzq)(1-b)\,\varphi_2 = 0. \tag{A.11}$$

$$cq(1-c)\,\varphi_1 - q(1-c)(c-abzq)\,\varphi_2 + bzq(c-aq)\,\varphi_3 = 0. \tag{A.12}$$

$$aq(1-c)\,\varphi_2 + (c-aq)\,\varphi_3 - c(1-aq)\,\varphi_4 = 0. \tag{A.13}$$

$$\varphi - \varphi_1 + \frac{(1-b)\,az}{1-c}\,\varphi_3 = 0. \tag{A.14}$$

$$cq(1-c)\varphi - q(1-c)(c-abzq)\varphi_2 + zq[ac+bc-abc-abq]\varphi_3 = 0. \tag{A.15}$$

$$(c-aq)(1-c)\varphi - (1-c)[c-aq+a^2zq(1-b)]\varphi_2 + z(1-aq)[ac+bc-abc-abq]\varphi_4 = 0. \tag{A.16}$$

$$aq(1-c)\varphi + [c-aq+a^2zq(1-b)]\varphi_3 - (1-aq)(c-abzq)\varphi_4 = 0. \tag{A.17}$$

$$(1-c)(1-cq)\varphi - (1-cq)[1-c-z\{a+b-ab(1+q)\}]\varphi_3 - z(1-aq)(1-bq)(c-abzq)\varphi_6 = 0. \tag{A.18}$$

$$aq(1-c)\varphi_1 + (c-aq)\varphi_3 + (1-aq)(c-abzq)\varphi_4 = 0. \tag{A.19}$$

$$(1-c)[cq(a+b)-abq^2(1+c)]\varphi_2 + (c-aq)(c-bq)\varphi_3 - c(1-aq)(1-bq)(c-abzq^2)\varphi_5 = 0. \tag{A.20}$$

$$\varphi_2 - \varphi_3 - \frac{(1-aq)(1-bq)\,cz}{(1-c)(1-cq)}\varphi_6 = 0. \tag{A.21}$$

$$\varphi_2 - \varphi_4 - \frac{(c-aq)(1-bq)z}{(1-c)(1-cq)}\varphi_6 = 0. \tag{A.22}$$

$$c\varphi_2 - (c-abzq^2)\varphi_5 - \frac{(c-aq)(c-bq)z}{(1-c)(1-cq)}\varphi_6 = 0. \tag{A.23}$$

Changing back to the original notation we have some more equations.

$$\Phi(a+) - \Phi(b+) - \frac{(a-b)z}{1-c}\,{}_2\varphi_1(aq, bq; cq; q, z) = 0. \tag{A.24}$$

$$\Phi - \Phi(zq) - \frac{(1-a)(1-b)z}{1-c}\,{}_2\varphi_1(aq, bq; cq; q, z) = 0. \tag{A.25}$$

$$\Phi - (1-a)\,\Phi(a+) - a\Phi(qz) = 0. \tag{A.26}$$

$$(1-c)\,{}_2\phi_1(a, b; c; q, z) + (c-b)\,{}_2\varphi_1(a, b; qc; q, qz) + (b-1)\,{}_2\phi_1(a, qb; qc; q, z) = 0. \tag{A.27}$$

## Acknowledgments

Parts of this work are in the Dissertation of C. Libis written under the supervision of M. E. H. Ismail at Arizona State University. The financial support of Arizona State University to C. Libis during his study is gratefully acknowledged. The authors are very grateful to David Masson of the University of Toronto and Mizan Rahman of Carleton University for their helpful comments and critical remarks on this work.

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